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Exact analysis of the spherical Raman–Nath equation

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Abstract. An exact solution of the spherical Raman-Nath equation is given. The result is applied to discuss photon statistics and squeezing properties of a free-electron laser.

1. Introduction

The spherical Raman-Nath equation (SRNE), originally derived to describe light diffraction by ultrasound [1], is important for discussing photon statistics and squeezing properties of a free-electron laser (FEL) [2]. In principle, SRNE is the following complicated differential-difference equation:

$$\mathrm{i}\frac{\mathrm{d}}{\mathrm{d}t} C_{l(t)}^{n_w, n_r} = (\lambda + \mu l + \nu l^2) C_{l(t)}^{n_w, n_r} + \Omega\{[(n_w - l)(n_r + l + 1)]^{1/2} C_{l+1(t)}^{n_w, n_r} + [(n_w - l + 1)(n_r + l)]^{1/2} C_{l-1(t)}^{n_w, n_r}\} \tag{1}$$

where λ, μ, ν and Ω are constant coefficients and n_w, $-n_r$ are up and down limits of the integer l, respectively. It is very difficult to solve SRNE exactly because of the existence of the non-linear term νl^2, so some perturbative theories [3-5] are used.

The purpose of the present paper is to solve SRNE exactly with the initial condition $C_{l(0)}^{n_w, n_r} = \delta_{l,0}$, which is profitable for discussing the higher non-linear effects of electron recoil in a FEL.

2. The exact solution of SRNE

One can analyse SRNE using a generalisation of the linear operational technique [6] which has been used in the special case of (1) with $\nu = 0$ [3]. The main procedures are as follows.

First, making the transformation

$$C_{l(t)}^{n_w, n_r} = (-\mathrm{i})^l \exp(\mathrm{i}\alpha x) \exp(\mathrm{i}\beta l x) \exp\{\mathrm{i}\gamma x[l + \tfrac{1}{2}(n_r - n_w)]^2\} |M_{l(x)}^{n_w, n_r}\rangle \tag{2}$$

where

$$x = \Omega t \tag{3}$$

$$\alpha = \frac{\nu(n_r - n_w)^2 - 4\lambda}{4\Omega} \tag{4}$$

$$\beta = \frac{\nu(n_r - n_w) - \mu}{\Omega} \tag{5}$$

$$\gamma = -\frac{\nu}{2\Omega} \tag{6}$$

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then defining a series of angular-momentum-type operators $\hat{L}_\pm$, $\hat{L}_z=\frac{1}{2}[\hat{L}_+,\hat{L}_-]$ as

$$\hat{L}_+|M^{n_w,n_r}_{l(x)}\rangle=[(n_w-l)(n_r+l+1)]^{1/2}|M^{n_w,n_r}_{l+1(x)}\rangle \tag{7}$$

$$\hat{L}_-|M^{n_w,n_r}_{l(x)}\rangle=[(n_w-l+1)(n_r+l)]^{1/2}|M^{n_w,n_r}_{l-1(x)} \tag{8}$$

$$\hat{L}_z|M^{n_w,n_r}_{l(x)}\rangle=[l+\tfrac{1}{2}(n_r-n_w)]|M^{n_w,n_r}_{l(x)}\rangle \tag{9}$$

and substituting (2)-(9) into (1), one obtains the following operational differential equation on $|M^{n_w,n_r}_{l(x)}\rangle$:

$$\frac{\mathrm{d}}{\mathrm{d}x}|M^{n_w,n_r}_{l(x)}\rangle=\exp(\mathrm{i}\gamma \hat{L}_z^2)[\mathrm{i}\gamma \hat{L}_z^2-\exp(\mathrm{i}\beta x)\hat{L}_++\exp(-\mathrm{i}\beta x)\hat{L}_-]\exp(-\mathrm{i}\gamma \hat{L}_z^2)|M^{n_w,n_r}_{l(x)}\rangle. \tag{10}$$

The non-linear term $\hat{L}_z^2$ appears on the right-hand side of (10). The special transformation must be made for one to solve equation (10):

$$|M^{n_w,n_r}_{l(x)}\rangle=\exp(\mathrm{i}\gamma x\hat{L}_z^2)\exp(-2h_{(x)}\hat{L}_z)\exp(g_{(x)}\hat{L}_+)\exp(-f_{(x)}\hat{L}_-|M^{n_w,n_r}_{l(0)}\rangle. \tag{11}$$

Inserting (11) into (10), we find that the three functions $f_{(x)}$, $g_{(x)}$ and $h_{(x)}$ happen to obey the set of equations (11) of [3] and the expressions (13) of [3] are still valid, only that β is defined by equation (5). Substituting the solutions of f, g and h into (11) and (2), one obtains

$$\begin{aligned}C^{n_w,n_r}_{l(t)}=&(-\mathrm{i})^l I^{n_w,n_r}_{l(x)}P^{l/2}_{(x)}(1-P_{(x)})^{(n_w-n_r-l)/2}\\&\times\exp\left\{\mathrm{i}(n_w-n_r-l)\tan^{-1}\left[\frac{\nu(n_r-n_w)-\mu}{\Omega\delta}\tan\left(\frac{\delta x}{2}\right)\right]\right.\\&\left.+\mathrm{i}\frac{x[\nu(n_w-n_r)^2+\mu(n_w-n_r)-2\lambda-l\nu(n_w-n_r)-2l\mu-l^2\nu]}{2\Omega}\right\}\end{aligned} \tag{12}$$

where

$$I^{n_w,n_r}_{l(x)}=\left(\frac{n_r!(n_r+l)!}{n_w!(n_w-l)!}\right)^{1/2}\sum_{j=0}^{n_r}\frac{(n_w+j)!(f_{(x)}g_{(x)})^j}{j!(l+j)!(n_r-j)!} \tag{13}$$

$$P_{(x)}=4\sin^2(\delta x/2)/\delta^2 \tag{14}$$

$$\delta=\left(4+\frac{[\nu(n_r-n_w)-\mu]^2}{\Omega^2}\right)^{1/2}. \tag{15}$$

The result (12) is the exact solution of SRNE which can be used in a large number of physical problems. In the following, particular emphasis is given to photon statistics and squeezing properties of a FEL.

3. Photon statistics of a FEL

For a helical pumped FEL, the symbols throughout the paper are summarised in table 1. The decay parameter λ is introduced because of spontaneous emission, which is analogous to an atomic laser. In the following, we take $\lambda=0$.

Assuming that the wiggler initial field is in the coherent state with a mean number of photons, $|\alpha_{w0}|^2$, and the laser initial field is vacuum which gives $n_r=0$, then

$$I^{n_w,n_r}_l=\binom{n_w}{l}^{1/2}. \tag{16}$$

Table 1.

c	speed of light
e	electron charge
m	electron mass
$\hbar = h/2\pi$	Planck constant
ε_0	dielectric constant of free space
V	interaction volume
P	electron axial momentum in the Bambini-Renieri frame
P_0	initial electron momentum in the Bambini-Renieri frame
$\omega = cK$	laser frequency in the Bambini-Renieri frame
$n_{r,w}$	photon numbers of laser (r) and wiggler (w)
l	number of exchanged photons
$C_l^{n_w,n_r}$	probability amplitude for interchange l photons in the presence of n_r laser and n_w wiggler photons
Λ	decay parameter
$\mu = -2KP_0/m$	resonance parameter
$\nu = 2\hbar K^2/m$	electron recoil parameter
$\Omega = e^2/2m\omega\varepsilon_0 V$	coupling constant

If the axial momentum of electron is positive in the Bambini-Renieri frame after the emission of n_w laser photons, i.e.

$$-2n_w\nu/\mu < 1 \tag{17}$$

then one can expand $p_{(x)}$ over ν (in the following, the expansion is to the second order of ν). Under the limits $|\alpha_{w0}|^2 \gg 1$, $\Omega \ll 1$, $|\mu/\Omega| \gg 1$ and $\Omega|\alpha_{w0}| = \text{constant } \bar{\Omega}$, one gets

$$\begin{aligned}\langle l\rangle = {} & \bar{\Omega}^2\left(\frac{\sin(\mu t/2)}{(\mu/2)}\right)^2 + \nu\bar{\Omega}^2\left(1+|\alpha_{w0}|^2\right)\frac{\partial}{\partial\mu}\left(\frac{\sin(\mu t/2)}{(\mu/2)}\right)^2 \\ & + \frac{2\nu^2\bar{\Omega}^2(\Omega^2+3\bar{\Omega}^2+\bar{\Omega}^2|\alpha_{w0}|^2)}{\mu^3}\frac{\partial}{\partial\mu}\left(\frac{\sin(\mu t/2)}{(\mu/2)}\right)^2 \\ & + \frac{\nu^2\bar{\Omega}^2(1+3|\alpha_{w0}|^2+|\alpha_{w0}|^4)}{2}\frac{\partial^2}{\partial\mu^2}\left(\frac{\sin(\mu t/2)}{(\mu/2)}\right)^2 + O(\nu^3)\end{aligned} \tag{18}$$

$$\begin{aligned}\Delta = {} & \langle l^2\rangle - \langle l\rangle^2 - \langle l\rangle \\ = {} & \nu\bar{\Omega}^4\frac{\partial}{\partial\mu}\left(\frac{\sin(\mu t/2)}{(\mu/2)}\right)^4 - \nu^2\bar{\Omega}^4|\alpha_{w0}|^2\left[\frac{\partial}{\partial\mu}\left(\frac{\sin(\mu t/2)}{(\mu/2)}\right)^2\right]^2 \\ & + \frac{2\nu^2\bar{\Omega}^4(3\Omega^2-2\bar{\Omega}^2)}{\mu^3}\frac{\partial}{\partial\mu}\left(\frac{\sin(\mu t/2)}{(\mu/2)}\right)^4 \\ & + \frac{\nu^2\bar{\Omega}^4(3-2|\alpha_{w0}|^2)}{2}\frac{\partial^2}{\partial\mu^2}\left(\frac{\sin(\mu t/2)}{(\mu/2)}\right)^4 + O(\nu^3)\end{aligned} \tag{19}$$

where

$$\langle l\rangle = \sum_{n_w=0}^{\infty}\sum_{l=0}^{n_w}\frac{\exp(-|\alpha_{w0}|^2)}{n_w!}|C_l^{n_w,n_r}|^2|\alpha_{w0}|^{2n_w}l$$

is the laser output,

$$\langle l^2\rangle = \sum_{n_w=0}^{\infty}\sum_{l=0}^{n_w}\frac{\exp(-|\alpha_{w0}|^2)}{n_w!}|C_l^{n_w,n_r}|^2|\alpha_{w0}|^{2n_w}l^2$$

is the second-order moment of photon numbers and Δ is the photon distribution.

To the first order of ν, the formula (19) presents: (i) sub-Poissonian (antibunching) for $\mu > 0$, (ii) Poissonian for $\mu = 0$, (iii) super-Poissonian (bunching) for $\mu < 0$. The linear gain is

$$G = \langle l \rangle - \langle l \rangle|_{\nu=0} = G_0 \frac{\mathrm{d}}{\mathrm{d}\theta}\left(\frac{\sin\theta}{\theta}\right)^2 \tag{20}$$

where $G_0 = \frac{1}{2}\nu\bar{\Omega}^2|\alpha_{w0}|^2 t^3$, $\theta = \frac{1}{2}\mu t$. The gain (20) has a positive sign in the $\theta < 0$ regime and its maximum, $0.54G_0$, occurs at $\theta = -1.3$.

To the second order of ν, the result (19) can be written in the explicit form:

$$\Delta = 8G_0 \sin^2\theta\left(\frac{\Omega}{\mu}\right)^2\left\{\left[1-4\xi\left(\frac{\Omega}{\mu}\right)^2\right]\frac{\mathrm{d}}{\mathrm{d}\theta}\left(\frac{\sin\theta}{\theta}\right)^2 - \xi\theta\frac{\mathrm{d}^2}{\mathrm{d}\theta^2}\left(\frac{\sin\theta}{\theta}\right)^2 - 6\xi\theta\left[\frac{\mathrm{d}}{\mathrm{d}\theta}\left(\frac{\sin\theta}{\theta}\right)\right]^2\right\} \tag{21}$$

where $\xi = \nu|\alpha_{w0}|^2/\mu$ and the formula (18) gives the non-linear gain

$$G = G_0\left\{\left[1+2\xi\left(\frac{\Omega}{\mu}\right)^2\right]\frac{\mathrm{d}}{\mathrm{d}\theta}\left(\frac{\sin\theta}{\theta}\right)^2 + \frac{\xi\theta}{2}\frac{\mathrm{d}^2}{\mathrm{d}\theta^2}\left(\frac{\sin\theta}{\theta}\right)^2\right\}. \tag{22}$$

The non-linear term

$$2\xi\left(\frac{\Omega}{\mu}\right)^2\frac{\mathrm{d}}{\mathrm{d}\theta}\left(\frac{\sin\theta}{\theta}\right)^2$$

means saturation for $\xi < 0$ because its sign is always opposite that of the linear gain; however, a gain enhancement is obtained for $\xi > 0$. In general, $|\xi| \ll 1$, so the maximum of (22), $0.54[1+2\xi(\Omega/\mu)^2]G_0$, occurs at $\theta_0 = -1.3(1-\frac{1}{2}\xi)$ and the other non-linear term

$$\frac{\xi\theta}{2}\frac{\mathrm{d}^2}{\mathrm{d}\theta^2}\left(\frac{\sin\theta}{\theta}\right)^2$$

is always positive at θ_0.

4. Squeezing properties of a FEL

As in [5], we can study the squeezing properties of a FEL. The expectation value of the electron longitudinal momentum is obtained as follows:

$$\langle p \rangle = \sum_{n_w=0}^{\infty}\sum_{l=0}^{n_w}\frac{\exp(-|\alpha_{w0}|^2)}{n_w!}|C_l^{n_w,n_l}|^2|\alpha_{w0}|^{2n_w}(p_0 - 2l\hbar K) = p_0 - 2\hbar K\langle l \rangle \tag{23}$$

and

$$\langle p^2 \rangle = \sum_{n_w=0}^{\infty}\sum_{l=0}^{n_w}\frac{\exp(-|\alpha_{w0}|^2)}{n_w!}|C_l^{n_w,n_l}|^2|\alpha_{w0}|^{2n_w}(p_0 - 2l\hbar K)^2 = p_0^2 - 4\hbar K p_0\langle l \rangle + 4\hbar^2 K^2\langle l^2 \rangle. \tag{24}$$

So the variance is

$$(\Delta p)^2 = \langle p^2 \rangle - \langle p \rangle^2 = 4\hbar^2 K^2\left[\bar{\Omega}^2 t^2\left(\frac{\sin\theta}{\theta}\right)^2 + G + \Delta\right] \tag{25}$$

To the second order of ν, the formula (25) gives

$$\left(\frac{\Delta p}{p_0}\right)^2 = 16|\alpha_{w0}|^2 \sin^2\theta \left(\frac{\Omega\nu}{\mu^2}\right)^2 . \tag{26}$$

The analytic expression (26) can be used to calculate the variance numerically for a range of parameters in table 1.

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